Universal entanglement of singular surfaces

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In general three-dimensional conformal field theories (CFTs), the entanglement entropy receives a logarithmic contribution whenever the entangling surface contains a sharp corner of opening angle θ. Such contribution is controlled by a regulator-independent function $a(\theta)$ which vanishes as $a(\theta) = \sigma\,(\theta - \pi)^2$ in the smooth-surface limit ($\theta \to \pi$). We review our recent conjecture that for general three-dimensional CFTs, this corner coefficient σ is determined by C_{T}, the coefficient appearing in the two-point function of the stress tensor through the relation $\sigma / C_{\mathrm{T}} = \pi^2/24$. We also comment on the extension of this conjecture to general Rényi entropies and higher-dimensional CFTs.

1 Introduction

Quantifying the low-energy effective number of degrees of freedom (dof) in conformal field theories (CFTs) is an interesting but challenging task, as these theories generally do not contain simple particle-like excitations. In two-dimensional CFTs, the Virasoro central charge provides a good measure of dof. It appears in many relevant quantities, like the entanglement entropy (EE) and the thermal free energy, and it is a renormalization group (RG) monotone [1]. In higher dimensions, quantum entanglement is becoming a fundamental tool to characterize interacting systems [2]. E.g., an analogous RG monotone for 3d CFTs was obtained from the EE of a disk-shaped region [3]. Here we shall focus however on a different measure of recent interest [4–28]: the coefficient capturing the contribution of sharp corners to spatial entanglement.

For any given quantum state, the EE of some spatial region V is defined as: $S = -\mathrm{Tr}\left(\rho_V \ln \rho_V\right)$, where ρ_V is the reduced density matrix resulting from integrating out the dof in the complement, $\overline{V}$. In the groundstate of a 3d CFT, the EE is given by: $S = B\,\ell/\delta - a(\theta)\,\ln(\ell/\delta) + \mathcal{O}(1)$, where ℓ is some characteristic length scale of V and δ is a short-distance cutoff, *e.g.,* the lattice spacing. The first, 'area law', term is regulator-dependent and scales with the size of the boundary. The second one appears only when V has a sharp corner with opening angle $\theta \in [0, 2\pi)$. Interestingly, $a(\theta)$ is a regulator-independent coefficient that captures well-defined physical information about the underlying CFT. It is positive and satisfies $a(2\pi - \theta) = a(\theta)$ for pure states [17]. Also, it behaves as:

$$a(\theta \to \pi) \simeq \sigma\,(\pi - \theta)^2\,, \tag{1}$$

in the limit of an almost smooth entangling surface. Prior to our studies in [4, 5], $a(\theta)$ had been studied for free fermions and scalars [17–19], interacting scalar theories via numerical simulations [20–22], Lifshitz quantum critical points [23], and holographic models [8, 24]. Altogether, the results in those papers suggest that $a(\theta)$ indeed provides an effective measure of the dof in the underlying CFT [19, 22].

Another quantity that quantifies the low energy dof is the central charge C_T, associated with the stress tensor $T_{\mu\nu}$ of the CFT. It characterizes the vacuum two-point function:

$$\langle T_{\mu\nu}(x)\, T_{\lambda\rho}(0)\rangle = \frac{C_T}{|x|^{2d}}\, I_{\mu\nu,\lambda\rho}(x)\,, \tag{2}$$

where $I_{\mu\nu,\lambda\rho}$ is a dimensionless tensor structure fixed by symmetry [29]. Most of this note will be devoted to reviewing our conjecture in [4, 5], which states that σ and C_T are related bye

$$\sigma / C_T = \pi^2/24\,, \tag{3}$$

for general CFTs. This is a somewhat surprising result since the EE is generally thought to be a nonlocal quantity, while our analysis indicates that the regulator independent corner contribution is controlled by a local correlation function in the nearly smooth limit (2).

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The remainder of this note goes as follows. In section 2, we review the calculation of σ/C_T for holographic theories dual to a family of higher-derivative bulk gravities and show how (3) is satisfied in all cases. In section 3 we verify that such relation is also satisfied for free massless scalars and fermions, which leads us to conjecture that (3) is valid for general CFTs in three dimensions. Finally, in section 4, we explain how our conjecture extends to general values of the Rényi index and to entangling surfaces consisting of (hyper)cones for higher-dimensional CFTs.

2 Holographic calculations

For holographic theories dual to Einstein gravity, EE is computed using the Ryu-Takayanagi prescription [30]:

$$S(V) = \underset{\gamma\sim V}{\mathrm{ext}}\left[\frac{\mathcal{A}(\gamma)}{4G}\right]. \tag{4}$$

Given a region V in the boundary CFT, we must find the codimension-2 surface in the bulk AdS spacetime which extremizes the area functional within the class of surfaces γ which are *homologous* to V on the asymptotic boundary. The EE is then obtained from the Bekenstein-Hawking formula $\mathcal{A}(\gamma)/(4G)$ evaluated on this extremal surface, where G is the Newton constant. This prescription for the holographic EE (HEE) was used to compute the corner coefficient $a_{\rm E}(\theta)$ for 3d CFTs dual to 4d AdS [8,24]; where 'E' indicates that the bulk theory is described by Einstein gravity. While $a_{\rm E}(\theta)$ is only implicitly known in terms of 2 integrals [24], it is nevertheless possible to compute the corner coefficient $\sigma_{\rm E}$ exactly. The result is: $\sigma_{\rm E} = \tilde{L}^2/(8\pi G)$, where $\tilde{L}$ is the radius of curvature of the AdS geometry [5].

$\sigma_{\rm E}$ — and more generally, $a_{\rm E}(\theta)$ — carries an overall factor of the ratio $\tilde{L}^2/G$ which is indicative of the number of dof in the boundary CFT — see *e.g.,* [31]. However, the very same ratio appears in many other physical quantities, *e.g.,* the stress tensor two-point function (2), EEs for general regions or the thermal entropy density. This is a simple manifestation of the fact that $\tilde{L}^2/G$ is the only dimensionless parameter in the bulk Einstein theory. As a consequence, all of these a priori distinct measures of the dof are all related in this class of CFTs. However, by introducing higher-curvature interactions, the bulk gravity acquires new dimensionless couplings and each of the above measures can acquire a distinct dependence on these couplings. The idea is then to determine whether the various measures are independent or if they encode the same information, *e.g.,* see [32–34].

In [4], we focused our attention on the following simple gravitational theory

$$I = \int \frac{d^4x\sqrt{g}}{16\pi G}\left[\frac{6}{L^2} + R + L^4\lambda_1 R\mathcal{X}_4 + L^6\lambda_2\mathcal{X}_4^2\right]. \tag{5}$$

The first two terms above are the (negative) cosmological constant with $\Lambda = -3/L^2$ and the standard Einstein term. The next two interactions are controlled by the dimensionless couplings $\lambda_{1,2}$ and contain $\mathcal{X}_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$, which is the Euler density on 4d manifolds. While $\mathcal{X}_4$ alone would be topological, $R\mathcal{X}_4$ and $\mathcal{X}_4^2$ are not, and so these terms do modify the gravitational equations of motion. In particular, AdS space (with radius of curvature $\tilde{L}$) is a solution of this theory provided: $1 = L^2/\tilde{L}^2 - 24\lambda_1 L^6/\tilde{L}^6 + 96\lambda_2 L^8/\tilde{L}^8$. In the following, we treat the higher-curvature interactions as perturbative corrections and we only calculate to leading order in $\lambda_{1,2}$.

When higher curvature interactions are considered, the Ryu-Takayanagi prescription (4) must be modified, see *e.g.,* [35]. Schematically, we have $S(V) = \underset{\gamma\sim V}{\mathrm{ext}}\, S_{\rm grav}(\gamma)$ where the new entropy functional contains the 'higher curvature corrections' to the Bekenstein-Hawking formula. Using the appropriate functional for the above higher curvature theory (5), one finds [4,5]

$$a(\theta) = \alpha\, a_{\rm E}(\theta) \quad \text{with} \;\; \alpha = 1 + 24\lambda_1 + \mathcal{O}(\lambda_i^2). \tag{6}$$

Hence the corner coefficient is modified by an overall factor but the θ-dependence is unchanged. The above result also implies $\sigma = \alpha\,\sigma_{\rm E}$.

Let us now compare this result to the central charge C_T. The stress tensor in the boundary CFT is dual to the metric perturbations in the bulk gravity theory, so (2) translates to a statement about the graviton propagator between two boundary points in AdS_4 [33]. Therefore we must determine the normalization of the graviton kinetic term in our higher-curvature theory (5). In a perturbative framework, it is straightforward to determine C_T by computing the equation of motion corresponding to the massless spin-2 graviton mode. In particular, we consider metric fluctuations $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is the background AdS_4 metric and $h_{\mu\nu} \ll 1$. The equation of the physical massless graviton can be isolated by choosing the transverse gauge condition $\bar{\nabla}^\mu h_{\mu\nu} = \bar{\nabla}_\nu h$, along with the tracelessness condition $\bar{g}^{\mu\nu}h_{\mu\nu} = 0$, and reads

$$-\frac{\alpha}{2}\left[\Box + \frac{2}{\tilde{L}^2}\right]h_{\mu\nu} = 8\pi G\, T^{bulk}_{\mu\nu}, \tag{7}$$

where α is the same coefficient as in (6). We have included the bulk stress tensor $T^{bulk}_{\mu\nu}$ of matter fields to establish the normalization of the graviton kinetic term. The effect of

α in (7) is to modify the holographic result for the stress tensor correlator (2) by an overall factor α:

$$C_T = (1+24\lambda_1 + \mathscr{O}(\lambda_i^2))\, C_{T,\mathrm{E}}\,, \tag{8}$$

where the Einstein result is $C_{T,\mathrm{E}} = 3\tilde{L}^2/(\pi^3 G)$ [33].

Remarkably then, for all the holographic CFTs dual to (5), we find that $a(\theta)/C_T = a_{\mathrm{E}}(\theta)/C_{T,\mathrm{E}}$, *i.e.*, we find a universal ratio that is independent of the details of the theory. Let us stress that this universality does *not* occur [5] when considering other measures of the boundary dof, such as the thermal entropy or the RG monotone F. In [5], this holographic analysis was extended to an 8-parameter family of higher curvature theories and it was found that the same universal ratio arises in the dual boundary CFTs.

3 Universal ratio

The universality revealed by our holographic analysis suggests more broadly that $a(\theta)/C_T$ provides a useful normalization if we wish to compare the corner coefficients of different field theories. This normalization was studied in [4,5], where we compared the holographic ratio $a(\theta)/C_T$ with the analogous ratios for free massless scalars and fermions, as well as for the Wilson-Fisher fixed points of the $O(N)$ models with $N = 1,2,3$. In the case of the free fields, $a(\theta)$ was computed for general values of θ by Casini and Huerta in [17–19], while C_T was evaluated in [29]: $C_T^{\mathrm{scalar}} = 3/(32\pi^2)$ and $C_T^{\mathrm{fermion}} = 3/(16\pi^2)$. On the other hand, the $O(N)$ model results for $a(\theta)$ have been so far restricted to the particular value $\theta = \pi/2$, and were obtained using numerical simulations for lattice Hamiltonians with the corresponding quantum critical points in [20–22]. C_T was recently determined for these models using conformal bootstrap methods [36].

Our analysis in [4,5] suggests that $a(\theta)/C_T$ is indeed an *almost univeral* ratio for all values of θ, in the sense that the values corresponding to all theories lie remarkably close to each other in the whole range of angles. We shall focus however in the almost smooth limit here.

The values of the corner coefficient σ for the free fields were computed up to some numerical precision in [17–19]: $\sigma_{\mathrm{scalar}} \simeq 0.0039063$, $\sigma_{\mathrm{fermion}} \simeq 0.0078125$. Using this, one finds: $(\sigma/C_T)_{\mathrm{scalar}} \simeq 0.411235$, $(\sigma/C_T)_{\mathrm{fermion}} \simeq 0.411234$. Hence, according to this result, the free field ratios agree with the holographic result, $\sigma_{\mathrm{E}}/C_{T,\mathrm{E}} = \pi^2/24 \simeq 0.411234$ with all the precision which is allowed by the numerical evaluation of the free field corner coefficients.

This motivated the conjecture (3) that the ratio $\sigma/C_T = \pi^2/24$ for all 3d CFTs. Our conjecture can be used to predict the *exact* free field values for σ, $\sigma_{\mathrm{scalar}} = 1/256\,, \quad \sigma_{\mathrm{fermion}} = 1/128$. To test this prediction, we revisited the original computations for these coefficients [17–19]. Improving the numerical accuracy in evaluating σ for the free fields, we found that the agreement between the numerical results and our prediction was easily extended to 1 part in 10^{12} – the accuracy to which we limited ourselves. Remarkably, soon after [4,5] appeared, the necessary integrals were evaluated analytically in [15], confirming our predictions. These two integrals are very complicated and not even similar. The fact that they yield simple fractions which differ by exactly a factor 2 is a highly non trivial and suggestive fact.

Given the extremely different nature of the CFTs involved in our comparison — holographic vs free field theories — and of the computations involved in evaluating $a(\theta)$ in each case, we conjectured that $\sigma/C_T = \pi^2/24$ is a universal result for all 3d CFTs. Remarkably, in [7,14], our conjecture was proven for general holographic theories whose gravity duals contain arbitrary contractions of the Riemann tensor. A proof for general CFTs is still missing though.

4 Generalizations

Entanglement entropy is a particular instance of a more general family of entanglement measures, the so-called Rényi entropies. These are defined as $S_n = 1/(1-n)\mathrm{Tr}\left(\rho_V^n\right)$, where n is a real-valued index and where the EE is recovered in the $n \to 1$ limit. General Rényi entropies also contain regulator independent corner functions $a_n(\theta)$, which generalize $a(\theta)$ and which in the smooth-surface limit behave as: $a_n(\theta \to \pi) = \sigma_n (\theta - \pi)^2$. In view of the fact that $\sigma_1 \equiv \sigma$ is completely characterized by C_T according to our conjecture in (3), it is natural to wonder what physical information is encoded in the analogous corner coefficients for $n \neq 1$.

In [6], we proposed that σ_n is related to the scaling dimension h_n of the so-called twist operators $\tau_n(V)$ — see *e.g.*, [37–40] for a detailed discussion of twist operators. In general dimensional theories, these are codimension-two operators extending along the entangling surface ∂V, which possess the defining property that their expectation value taken in the symmetric product of n-copies of the original CFT yields $\langle \tau_n(V)\rangle_n = \mathrm{Tr}\left(\rho_V^n\right)$. Their scaling dimension is in turn defined by the coefficient of the leading power-law divergence in the correlator $\langle T_{\mu\nu}\tau_n\rangle_n$ as the distance y between both operators decreases. Schematically we have: $\langle T_{\mu\nu}\tau_n\rangle_n = -h_n/(2\pi y^d)\, t_{\mu\nu} + \cdots$, for some fixed tensorial structure $t_{\mu\nu}$ [6, 39, 40]. Importantly, h_n does not depend on the geometric details of ∂V so, just like C_T, it is a quantity which characterizes each theory.

In [6], we proposed the following relation between these scaling dimensions and the corner coefficients for general CFTs,

$$\sigma_n = \frac{h_n}{\pi(n-1)}\,. \tag{9}$$

This reduces to our original conjecture for $n=1$ by virtue of an interesting relation [40], relating $\partial_n h_n|_{n=1}$ to C_T for general CFTs. Similarly, it gives the right answers for free fermions and scalars for general values of n [6, 9, 10].

Another straightforward extension of our results consists of considering (hyper)conical entangling surfaces in higher-dimensional CFTs. The corresponding regulator-independent contributions to the Rényi entropies are controlled by certain functions $a_n^{(d)}(\theta)$ which again behave as $a_n^{(d)}(\theta\to\pi)=\sigma_n^{(d)}(\theta-\pi)^2$ in the analogous smooth limits.

In [7], we showed that the two conjectures (3) and (9) valid for three-dimensional CFTs can be extended to a more general relation valid for such (hyper)conical contributions. In particular, we propose the following formula

$$\sigma_n^{(d)} = \frac{h_n}{n-1}\,\frac{(d-1)(d-2)\,\pi^{\frac{d-4}{2}}\,\Gamma\left[\frac{d-1}{2}\right]^2}{16\,\Gamma[d/2]^3} \times \begin{cases} \pi & d \text{ odd}, \\ 1 & d \text{ even}, \end{cases}$$

where h_n is the scaling dimension of the corresponding twist operator. This result — which we have proven to apply for general holographic theories for $n=1$ — reduces to our previous conjectures in the obvious limits, and correctly reproduces all the known results (both for $n=1$ and $n\neq 1$) for higher-dimensional theories, including broad classes of $d=4$ and $d=6$ CFTs [7, 41, 42].

Key words. AdS/CMT, Quantum entanglement, Conformal field theories.

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